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TREES IN REGULAR GRAPHS AND DOUBLE CORRESPONDENCE SYSTEMS

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by

E. Wattel

Abstract

In this paper we investigate the double correspondence graph problem of Koopman. This is a probabilistic variant of the well-known problems of maximal systems of distinct representatives and optimal matchings. It is treated by the originator, by Finch and by Marshall from the point of view of operations research, but in this paper we use more combinatorial and analytic methods to describe the infinite solutions of the problem. After the preliminaries we enumerate the number of small subtrees through a given vertex in a regular graph. From the coefficients we construct a power series which provides the solution of the double correspondence problem for infinite regular trees. From this infinite solution we derive the bounds for the finite graph cases.

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1. Double correspondence systems

Preceding the main definition of the system and the first observations we describe a practical situation for which it is a mathematical model.

Let us consider a situation in which N installations require service from A service stations. Every service station can deliver service to precisely two installations,*) and every installation can receive service of precisely v service stations. (Obviously $A=\frac{Nv}{2}$). Suppose next that at a certain time each service station independently has a probability p of being unable to deliver service and that no service station can deliver service to more than one installation at the same time.

Now we ask for the expected number of installations which can get service, and how the service stations should be distributed over the installations such that the expected percentage of the installations which gets service, is as large as possible.

1.1. Definition of the problem

Let G be a regular graph without loops with degree v, which has N vertices and A edges. Suppose that a subcollection of A is removed from G subject to the following conditions:

i) every edge of G has a fixed probability p of being removed;

ii) the removal of each edge is stochastically independent of the removal of the other edges.

Then we get a *reduced graph* G_r.

In G_r we construct an *optimal edge-vertex assignment* (in the sequel briefly an *assignment*), this is a one to one function on a subcollection

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*)The problem in which every service station can deliver service to another number of installations is called the multiple correspondence problem ([2],[4]). Only the double correspondence problem is related to regular graphs. The multiple correspondence problem can be described with bipartite graphs.

of the edges into the collection of vertices of the reduced graph, which maps each edge onto one of its vertices, and which has a range as large as possible under these conditions.

A vertex which is not in the range of an optimal assignment is called an *excluded vertex*. Let N_r be the number of excluded vertices in G_r. Then the expectation $\mathcal{E}(N_r/N)$ is a stochastic function which depends on the graph G and the probability p. We call this the *service polynomial* $F_G(p)$. It is the expected fraction of the vertices which are not represented by edges in an optimal assignment.

The double correspondence problem asks for minimization of the polynomial $F_G(p)$ by variation of G over regular graphs with a fixed degree under certain restrictions for the number of vertices N and the edge removal probability p. Usually both N and p are fixed.

Our aim is to construct the greatest lower bound function for the polynomials $F_G(p)$ of all regular graphs with degree v by means of the service function F(p) of the infinite regular tree of degree v.

1.2. Notation conventions and definitions

The probability that an edge is not removed is denoted by q ($q = 1-p$); therefore $0 \le q \le 1$ since $1 \ge p \ge 0$. $F^*_G(q) = F_G(1-q)$ and $F^*(q) = F(1-q)$.

An isolated vertex is a tree without edges. An n-(sub)tree is a (sub)tree with n edges ($n \ge 0$).

Let S be a subgraph of G. An edge of G which is not in S, but which is incident with a vertex of S is called *peripheral* to S.

1.3. Proposition

In every optimal assignment in a (reduced) graph G_r each excluded vertex is contained in a finite tree component of G_r and each finite tree component contains precisely one excluded vertex.

Proof. In a finite tree component the number of edges is one less than the number of vertices. Therefore an optimal assignment in it contains at least one excluded vertex. We can construct an optimal assignment by excluding one vertex and directing all edges inward toward the excluded vertex. Next we assign each edge to its initial vertex and in this way precisely one vertex is excluded from the assignment.

In a finite non-tree component of G_r we construct an assignment without excluded vertex by taking a spanning tree of the component and making an assignment as above in this tree, with one of the vertices of a removed edge a_0 as the excluded vertex. Finally, we assign a_0 to this vertex and now each vertex is contained in the assignment. Therefore no optimal assignment can have excluded vertices in this component.

In an infinite component of G_r no optimal assignment will have excluded vertices either. We restrict our attention to a spanning tree T of the component. A theorem of König ([3]; VI Satz 4) states that in an infinite tree there exists a one to one correspondence between the edges and the vertices. Such a correspondence in T can be considered as an optimal assignment in the component of G_r.

From these three cases the proposition is clear. □

1.4. Proposition

If T is a finite subtree of a graph G then the probability of the event that T is a component of G_r is equal to $p^{m(T)}.q^{n(T)}$ in which $n(T)$ is the number of edges of T and $m(T)$ is the number of peripheral edges to T.

This proposition follows from the independence of the removal of the edges (1.1, condition ii).

1.5. Proposition

The service polynomial of a finite graph G is

$$F_G(p) = \frac{1}{N} \sum_{T} p^{m(T)} q^{n(T)},$$

where the sum is taken over all subtrees of G, and $m(T)$ and $n(T)$ are defined as in 1.4.

Proof. By definition

$$F_G(p) = \frac{1}{N} \sum_{G_r}^{*} q^{A_r} p^{A-A_r} \sum_{T\subset G_r}^{**} 1,$$

in which $\sum^*$ is taken over all reduced graphs G_r of G, and A_r is the number of edges G_r, and the sum $\sum^{**}$ is taken over all tree components of G_r. Reversal of the order of summation yields

$$\frac{1}{N} \sum_{T} \sum_{G_r}^{***} q^{A_r} p^{A-A_r}$$

in which $\sum^{***}$ is taken over all reduced graphs which contain T as a component. Now for a fixed T

$$\sum^{***} q^{A_r} p^{A-A_r} = p^{m(T)} q^{n(T)} ,$$

and the assertion follows. □

2. Trees in regular graphs

In this section we enumerate the trees, which pass through a given vertex in regular graphs with less edges than the girth of the graph. To this end we define generating functions and we use Lagrange's inversion formula in order to obtain the coefficients of the related power series.

2.1. Proposition

Let G be a regular graph with girth g and degree v. Let T be a subtree of G with $n < g-1$ edges. Then there are $(v-2)n + v$ peripheral edges to T and the probability that T is a component of G_r is $p^v.(p^{v-2}q)^n$.

Proof. T has n+1 vertices. Each vertex is incident with v edges of G, so there are $(n+1)v$ incidence relations involved. Each edge of T is incident with two vertices and since $n < g-1$ no addition of an edge can yield a cycle. Therefore no peripheral edge can be incident with two vertices of T. We obtain $m(T) = (n+1)v - 2n$. The proposition now follows from 1.4. □

2.2. Theorem

Let G be a regular graph with degree v and girth g. Then for each $n < g$ and each vertex X of G there are

$$B_n = \frac{v}{(v-2)n + v} \binom{(v-1)(n+1)}{n}$$

n-subtrees of G, which contain X as a vertex. For $n \geq g$ the number B_n is an upper bound for the number of n-trees.

Proof. We first enumerate all rooted trees with root X and with a fixed initial edge a_j. Let K_k be the number of k-trees of this type. Then $K_0 = 1$ (no edges) and $K_1 = 1$ (the edge a_j) and

$$K_k = \sum{}^{*} (K_{i1} \cdot K_{i2} \cdot \ldots \cdot K_{i(v-1)}) \qquad (2.1)$$

where the sum is taken over all sequences of v-1 non-negative integers with sum k-1. This is a consequence of the fact that we can construct trees emerging from the other vertex of a_j on a fixed edge adjacent to a_j which are structurally equivalent to the trees emerging from X with initial edge a_j. Define

$$\Psi(z) = \sum_{k=0}^{\infty} K_k z^k \tag{2.2}$$

then, if we consider (2.1) as power series multiplication, we find

$$(\Psi(z))^{v-1} = \frac{\Psi(z)-1}{z}, \tag{2.3}$$

or, if

$$\Psi(z) -1 = \psi, \tag{2.4}$$

then

$$z = \psi(1+\psi)^{1-v} \tag{2.5}$$

Next we apply Lagrange's inversion formula (cf.[1]) to (2.2) and (2.5)

$$\psi = \sum_{k=1}^{\infty} K_k z^k$$

with

$$K_k = \frac{1}{k!} \left(\frac{d}{d\psi}\right)^{k-1} \left\{(\psi+1)^{k(v-1)}\right\}\Big|_{\psi=0} = \frac{1}{k(v-2)+1} \cdot \binom{k(v-1)}{k} \tag{2.6}$$

In order to find all n-trees containing X we combine trees emerging from the edges $a_1,\dots,a_v$ incident with X such that the sum of the edges of the trees emerging from the edges $a_1,\dots,a_v$ of X is equal to n. In the same way as (2.1) we obtain

$$B_n = \sum{}^{*}(K_{j1} \cdot K_{j2} \cdot \dots \cdot K_{jv}), \tag{2.7}$$

in which the sum is taken over all v-tuples $j_1, j_2, \ldots, j_v$ with sum n. Analogous to (2.2) and (2.3) we obtain

$$\Phi(z) = \sum_{n=0}^{\infty} B_n z^n = (\Psi(z))^v. \tag{2.8}$$

In order to find the coefficients of these series we introduce another variable q defined as the solution in the neighbourhood of 0 of the equation

$$q(1-q)^{v-2} = z \tag{2.9}$$

(cf. section 1.2 and proposition 2.1). Another application of Lagrange yields

$$q = \sum_{k=1}^{\infty} \gamma_k z^k$$

with

$$\gamma_k = \frac{1}{k!} \left(\frac{d}{dq}\right)^{k-1} \{(1-q)^{-k(v-2)}\}\Big|_{q=0} = \frac{1}{k!} \frac{\{k(v-1)-2\}!}{\{k(v-2)-1\}!}, \tag{2.10}$$

and

$$(1-q)^{-1} = 1 + \sum_{k=1}^{\infty} \delta_k z^k$$

with

$$\delta_k = \frac{1}{k!} \left(\frac{d}{dq}\right)^{k-1} \left\{\frac{d(1-q)^{-1}}{dq} (1-q)^{-k(v-2)}\right\}\Big|_{q=0} =$$

$$= \frac{1}{k!} \left(\frac{d}{dq}\right)^{k-1} \{(1-q)^{-k(v-2)-2}\}\Big|_{q=0}$$

we obtain

$$\delta_k = K_k = \frac{\{k(v-1)\}!}{k!\{k(v-2)+1\}!} \tag{2.11}$$

and therefore we conclude that

$$\Psi(z) = (1-q)^{-1} \quad \text{and} \quad \Phi(z) = (1-q)^{-v}. \tag{2.12}$$

Now we can again apply Lagrange on $\Phi(z)$ in order to obtain the coefficients B_n; since $\Phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$ we find

$$B_n = \frac{1}{n!} \left(\frac{d}{dq}\right)^{n-1} \left\{\frac{d(1-q)^{-v}}{dq} (1-q)^{-n(v-2)}\right\}\Big|_{q=0} =$$

$$= \frac{1}{n!} \left(\frac{d}{dq}\right)^{n-1} \left\{\frac{v}{(1-q)^{(n+1)(v-2)+3}}\right\}\Big|_{q=0} = \frac{v}{n!} \frac{\{(n+1)(v-1)\}!}{\{(n+1)(v-2)+2\}!}$$

and the required formula for B_n follows immediately.

This way of generating trees is only valid as long no cycle is introduced in the procedure. This is the case when the original graph is an infinite regular tree and when the number of edges involved in the computation is smaller than g.

B_n is completely determined by the K_k with $k \leq n$ and also the enumeration of K_k uses only trees with less than k edges. Therefore B_n is indeed the number of n-trees containing X when $n < g$.

When $n \geq g$ the only thing which can happen is that there are combinations of K's in (2.1) and (2.7) which do not represent trees. In that case the coefficient B_n will be larger than the actual value of the number of n-trees containing X. This finishes the proof. □

For the infinite regular trees G the number B_n is the number of n-subtrees through a vertex of G for every number n.

3. The infinite regular tree

In this section we consider the infinite regular tree of a given degree v and we will derive an expression for the service function F(p) of this graph. This service function has to be defined as the probability that a given vertex is excluded from an optimal assignment. Since exclusion is a choice process we should make an assumption about it such that this choice is independent of the particular vertex.

3.1. Assumption

Let X be a vertex of an n-tree T with probability $\mathcal{E}(T)$. We define the probability that X is excluded from service under the condition that T is its component in the reduced graph to be $\frac{1}{n+1}$. The contribution to the loss of service of X because of the occurence of T will therefore be $\mathcal{E}(T)/(n.1)$.

3.2. Proposition

The service function $F(p) = F^*(q)$ of an infinite regular tree with degree v is equal to

$$F(p) = p^v \sum_{n=0}^{\infty} \frac{B_n}{n+1} (p^{v-2}q)^n = vp^v \sum_{n=0}^{\infty} \frac{\{(n+1)(v-1)\}!}{(n+1)!\{(n+1)v-2n\}!}(p^{v-2}q)^n.$$

Moreover, this series converges for $0 \leq p \leq 1$ and $q = 1-p$ absolutely and uniformly and the limit function is analytical on $0 \leq p < (v-2)/(v-1)$ and on $(v-2)/(v-1) < p \leq 1$ and continuous in $p = (v-2)/(v-1)$.

Proof. The formula for F(p) is a direct consequence of the assumption 3.1, proposition 2.1 and theorem 2.2. In order to establish convergence we only have to consider the radius of convergence of $\sum_{n=0}^{\infty} \frac{B_n}{n+1} z^n$.

(cf.[6], formula A2)

$$\frac{B_n}{n+1} = \binom{(n+1)(v-1)}{n+1} \frac{1}{\{(n+1)v-2n\}\{(n+1)v-2n-1\}} \leq$$

$$\leq \sqrt{\frac{v-1}{2\pi(n+1)(v-2)}} \left(\frac{(v-1)^{v-1}}{(v-2)^{v-2}}\right)^{n+1} \frac{1}{\{(n+1)v-2n-1\}\{(n+1)v-2n\}} \leq$$

$$\leq \left(\frac{(v-1)^{v-1}}{(v-2)^{v-2}}\right)^{n+1} \left(\frac{1}{n}\right)^{5/2} .$$

Therefore $\sum_{n=0}^{\infty} \frac{B_n z^n}{n+1}$ is an analytic function which is convergent on the closed disc $|z| \leq (v-2)^{v-2}\,(v-1)^{1-v}$.

If we put $z = p^{v-2}(1-p)$ then z is an analytic function of p with a unique absolute maximum for $p = (v-2)/(v-1)$ in the considered interval. There z is equal to the maximal admissible value $(v-2)^{v-2}\,(v-1)^{1-v}$. This proves the proposition. □

3.3. Theorem

The service function $F(1-q) = F^*(q) = 1 - \frac{1}{2}vq$ for $0 \leq q \leq \frac{1}{v-1}$.
The service function

$$F(p) = \frac{p^v(1-\frac{1}{2}vy)}{(1-y)^v} \text{ for } 0 \leq p \leq \frac{v-2}{v-1} ,$$

in which y is the unique solution of the equation $p^{v-2}(1-p) = y(1-y)^{v-2}$ with $0 \leq y \leq \frac{1}{v-1}$.

Proof. We consider the sum $\sum_{n=0}^{\infty} \frac{B_n z^n}{n+1}$. In order to prove the first formula we must show that this is equal to $(1-\frac{1}{2}vq)(1-q)^{-v}$, whenever $z = q(1-q)^{v-2}$.

We again apply the Lagrange formula on these equations as in the proof of theorem 2.2. We have

$$(1-\tfrac{1}{2}vq)(1-q)^{-v} = \sum_{n=1}^{\infty} F_n\, z^n + 1,$$

in which

$$F_n = \frac{1}{n!}\left(\frac{d}{dq}\right)^{n-1} \left\{\frac{v\{1-(v-1)q\}}{2(1-q)^{v+1}} \cdot \frac{1}{(1-q)^{n(v-2)}}\right\}\Big|_{q=0} =$$

$$= \frac{v}{2}\left\{\frac{1}{n!}\,\frac{\{(n+1)(v-1)\}!}{\{(n+1)(v-2)+2\}!} - \frac{(n-1)(v-1)}{n!}\,\frac{\{(n+1)(v-1)-1\}!}{\{(n+1)(v-2)+2\}!}\right\} =$$

$$= \frac{v}{2}\frac{\{(n+1)(v-1)\}!}{(n+1)!\{(n+1)(v-2)+2\}!}\{(n+1)-(n-1)\} = \frac{B_n}{n+1}.$$

Since $B_0 = 1$ we find that in a neighbourhood of 0 the functions $F^*(q)(1-q)^{-v}$ and $(1-\frac{1}{2}vq)(1-q)^{-v}$ are equal.

However, we know that $F^*(q)$ is analytical on the interval $0 \le q < 1/(v-1)$ (cf.3.2) and therefore we have $F^*(q) = 1 - \frac{1}{2}vq$ on $0 \le q \le 1/(v-1)$. The limit in $\frac{1}{v-1}$ follows from the continuity which was also established in proposition 3.2.

In order to find the formula for $0 \le p \le (v-2)/(v-1)$ we observe that $p^{v-2}(1-p)$ has a unique point with derivative 0 inside the interval $0 < p < 1$, namely in $p = (v-2)/(v-1)$. The function $z(p) = p^{v-2}(1-p)$ is strictly increasing from 0 to $(v-2)/(v-1)$ and strictly decreasing from $(v-2)/(v-1)$ to 1. Therefore for each p in $(0,(v-2)/(v-1))$ there exists a unique y with $y(1-y)^{v-2} = p^{v-2}(1-p)$ subject to $0 \le y \le 1/(v-1)$.

From the first part of the proof we find:

$$\sum_{n=0}^{\infty} \frac{B_n}{n+1} z^n = (1-\tfrac{1}{2}vy)(1-y)^{-v}$$

and our second formula is a direct corollary of this result. It is also clear that this formula is analytical and a direct computation of the derivatives shows that $F(p)$ is even differentiable in $p = (v-2)/(v-1)$. □

3.4. Corollaries

a) If $v = 3$ then $F(p) = \dfrac{(1-\frac{3}{2}p)p^3}{(1-p)^3}$ for $0 \le p \le \frac{1}{2}$.

b) If $v = 4$ then $F(p) = \dfrac{16p^4(\sqrt{1+2p-3p^2}-p)}{(1-p+\sqrt{1+2p-3p^2})^4}$ for $0 \le p \le \frac{2}{3}$.

Proof.

a) $y(1-y) = p(1-p)$ if $v = 3$. So $p = y$.

b) $y(1-y)^2 = p^2(1-p)$. Therefore $y^3-2y^2+y+p^3-p^2 = 0$; and so

$(y+p-1)(y^2+p^2-py-y) = 0$. The first factor vanishes when $y = q = 1-p$. The second factor vanishes when $y = \frac{1}{2}(p+1\pm\sqrt{1+2p-3p^2})$. We have to choose the negative root since $y \le \frac{1}{3}$ and we thus get

$$F(p) = \frac{16p^4(\sqrt{1+2p-3p^2}-p)}{(1-p+\sqrt{1-2p-3p^2})^4}. \quad \square$$

From these formulas we can give some numerical values:

a) $v = 3$, $p = \frac{1}{2}$; then $F(p) = \frac{1}{4}$. Notice, that we have for the graph K_4, $F_G(\frac{1}{2}) = \frac{9}{32} > \frac{1}{4}$.

b) $v = 4$, $p = \frac{1}{2}$; then $F(p) = 0{,}0902$. If we consider K_5, we find $F_G(\frac{1}{2}) = 0{,}130 > 0{,}0902$.

The boundaries given in [2] for $F_G(\frac{1}{2})$ in case $v = 4$ are for large graphs G with "optimal" structure: $0{,}084 \le F_G(\frac{1}{2}) \le 0{,}125$. Marshall [5] finds $0{,}084 \le F_G(\frac{1}{2}) \le 0{,}106$ (we have adapted the notation).

4. Estimates for finite graphs

In this section we prove that the value of $F(p)$ is always smaller than the value of $F_G(p)$ for each finite graph G and for every p and v.

We also show that for p close to 0 and for p close to 1, the function $F(p)$ is the uniform greatest lower bound of the $F_G(p)$. The polynomial $F_G(p)$ will be close to $F(p)$ whenever the girth is large.

4.1. Lemma

For $p = 1-q$ and $0 \le q \le 1$ the following equality holds:

$$\sum_{i=0}^{\frac{1}{2}vN} (N-i)\binom{\frac{1}{2}vN}{i} p^{\frac{1}{2}vN-i}\, q^{i} = N(1-\tfrac{1}{2}vq).$$

Proof.

$$(p+q)^{\frac{1}{2}vN} = \sum_{i=0}^{\frac{1}{2}vN} \binom{\frac{1}{2}vN}{i} p^{\frac{1}{2}vN-i}\, q^{i}. \tag{4.1}$$

Differentiation and multiplication with q gives:

$$\tfrac{1}{2}vN\; q(p+q)^{\frac{1}{2}vN-1} = \sum_{i=0}^{\frac{1}{2}vN} i\binom{\frac{1}{2}vN}{i}\, p^{\frac{1}{2}vN-i}\, q^{i}. \tag{4.2}$$

If we multiply (4.1) with N and subtract (4.2) we find

$$N(p+q)^{\frac{1}{2}vN} - \tfrac{1}{2}vN\; q(p+q)^{\frac{1}{2}vN-1} = \sum_{i=0}^{\frac{1}{2}vN} (N-i)\binom{\frac{1}{2}vN}{i}\, p^{\frac{1}{2}vN-i}\, q^{i}.$$

Now the lemma follows from $p+q = 1$. □

4.2. Theorem

If G is a finite regular graph of degree v then for all p we have $F_G(p) \ge F(p)$ in which $F(p)$ is the service function of the infinite regular tree of degree v.

Proof. A reduced graph with i edges has at least N-i excluded vertices. Therefore

$$F_G(p) \geq \frac{1}{N} \sum_{i=0}^{N} (N-i)\binom{\frac{1}{2}vN}{i} q^i p^{\frac{1}{2}vN-i} \geq \frac{1}{N} \sum_{i=0}^{\frac{1}{2}vN} (N-i)\binom{\frac{1}{2}vN}{i} q^i p^{\frac{1}{2}vN-i} = \tag{4.3}$$

$$= 1 - \tfrac{1}{2}vq = F^*(q)$$

whenever $0 \leq q \leq \frac{1}{1-v}$. In this formula the second inequality holds because all the extra terms are negative.

Next we assume that $0 \leq p \leq (v-2)/(v-1)$. Let y and z be such that $z = y(1-y)^{v-2} = p^{v-2}(1-p)$ and $0 \leq y \leq (v-1)^{-1}$. Now

$$\frac{F(p)}{p^v} = \sum_{n=0}^{\infty} \frac{B_n}{n+1} z^n = \frac{F^*(y)}{(1-y)^v} \tag{4.4}$$

$F_G(p) = \frac{1}{N} \sum_{T \text{ subtree of } G} p^{m(T)} q^{n(T)}$ in which $n(T)$ is the number of edges of the tree and $m(T)$ is the number of peripheral edges to the tree. Now $m(T) \leq n(T)(v-2) + v$. Define $d(T) = n(T)(v-2) + v - m(T)$.(This is the number of peripheral edges which have both vertices in the tree.) Then

$$F_G(p) = \frac{1}{N} \sum_{T \text{ subtree of } G} p^{v-d(T)}(p^{v-2}q)^{n(T)}. \tag{4.5}$$

Now it follows that

$$\sum_{n=0}^{\infty} \frac{B_n z^n}{n+1} = \frac{F^*(y)}{(1-y)^v} \leq \frac{F_G^*(y)}{(1-y)^v}$$

(compare with (4.3)). Moreover,

$$\frac{F_G(1-y)}{(1-y)^v} = \frac{1}{N} \sum_T \frac{((1-y)^{v-2}y)^{n(T)}}{(1-y)^{d(T)}} \leq \frac{F_G(p)}{p^v} \quad ,$$

since $1-y \leq p$ and $d(T)$ is always positive (cf.(4.5)). We find

$$\frac{F(p)}{v} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n+1} = \frac{F(1-y)}{(1-y)^v} \leq \frac{F_G(1-y)}{(1-y)^v} \leq \frac{F_G(p)}{p^v}$$

and now the assertion follows easily. Both parts together prove the theorem. □

4.3. Theorem

Let $\{G_g | g = 2,3,4,\ldots\}$ be a sequence of regular graphs *) each with degree v and girth g. Then $F_{G_g}(p)$ tends to $F(p)$ whenever $g \to \infty$ and p is such that

$$p^{\frac{1}{2}v-1}(1-p) \leq \frac{(v-2)^{v-2}}{(v-1)^{v-1}} .$$

Proof. We consider the tree representation of $F_{G_g}(p)$ and divide it into two parts:

$$F_{G_g}(p) = \frac{1}{N_g} \sum_{\substack{T \\ n(T)<g-1}} p^{m(T)} q^{n(T)} + \frac{1}{N_g} \sum_{\substack{T \\ n(T)\geq g-1}} p^{m(T)} q^{n(T)} .$$

In the first part we know that $m(T) = (v-2)n(T)+v$ and therefore this part is equal to the first g-1 terms of the sum series for F(p) (cf. proposition 2.1 and theorem 2.2). Since g tends to ∞ this part converges to F(p).

It remains to prove that the second part tends to 0. From theorem 2.2 we know that the number of n-trees through a vertex in a graph is not larger than the number of n-trees through a vertex in the corresponding infinite regular tree. One n-tree has n+1 vertices and therefore the number of trees with n edges is $\frac{1}{n+1} B_n N_g$. An n-tree has (v-2)n+v peripheral incidence relations, and therefore at least $\frac{(v-2)n+v}{2}$ peripheral edges. Now we estimate the second sum:

*) From the theorem of Erdös and Sachs [7] it follows that there exist sequences $\{G_g\}$ of this type.

$$\frac{1}{N_g} \sum_{\substack{T \\ n \geq g-1}} p^{m(T)} q^n \leq \sum_{n=g-1}^{\infty} \frac{B_n}{n+1} q^n p^{(\frac{1}{2}v-1)n} p^{\frac{1}{2}v} \leq$$

$$\leq p^{\frac{1}{2}v} \sum_{n=g-1}^{\infty} \frac{B_n}{n+1} \left\{ \frac{(v-2)^{v-2}}{(v-1)^{v-1}} \right\}^n$$

The last series is convergent for all g and hence it tends to 0 whenever g tends to ∞ . □

4.4. Theorem

Let G be a regular graph with degree v and girth g. Then we have

$$F_G^*(q) = 1 - \tfrac{1}{2} vq + O(q^{1\frac{1}{2}g}) \qquad \text{for } q \downarrow 0.$$

Proof. A reduced graph with less than $\frac{3}{2}g$ edges cannot contain two independent circuits and therefore every edge is in the domain of an assignment and the number of excluded vertices is then equal to N-n. We find

$$F_G^*(q) = \sum_{n=0}^{1\frac{1}{2}g-\frac{1}{2}} \frac{(N-n)}{n} q^n (1-q)^{\frac{1}{2}vN-n} \binom{\frac{1}{2}vN}{n} + O(q^{\frac{2}{3}g}).$$

The theorem now easily follows from 4.1. □

4.5. Theorem

Let G be a graph with degree v and girth g. Then we have

$$F_G(p) = F(p) + O(p^{\frac{1}{2}g(v-2)+1}) \qquad \text{for } p \downarrow 0.$$

Proof.

$$F_G(p) = \frac{1}{N} \sum_{\substack{T \\ n(T)<g-1}} p^{m(T)} q^{n(T)} + \frac{1}{N} \sum_{\substack{T \\ n(T)\geq g-1}} p^{m(T)} q^{n(T)}.$$

In theorem 4.3 we have already seen that the first part of this sum is equal to the first part of $F(p)$ up to the order $p^{(g-2)(v-2)+v}$. We still have to prove that the second part is $O(p^{\frac{1}{2}(v-2)g})$. If $n(T) \geq g-1$ then $m(T) \geq \frac{n(T)(v-2)+v}{2} \geq \frac{g(v-2)}{2} + 1$. This proves the theorem. □

4.6. Conclusions

The last three theorems show the close relation between the double correspondence problem and the theorem of Erdös and Sachs (cf.[7]) about regular graphs with large girth. Since there exist graphs with arbitrary large girth we conclude that in the intervals mentioned in theorem 4.3 the functions $F_{G_g}(p)$ tend to $F(p)$; i.e. on those intervals $F(p)$ is the greatest lower bound for the functions $F_G(p)$ in which the $F_G(p)$ correspond to the finite regular graphs G of degree v.

The last theorem can be improved in the following way:

$$F_G(p) = F(p) + O(p^{g(v-2)}) \qquad \text{for } p \downarrow 0 ,$$

but

$$\{F_G(p) - F(p)\}\ p^{-g(v-2)-1}$$

has a limit for p tends to 0, for every graph with girth g. This is a corollary of the fact that every n-tree in a regular graph with $n \geq g > 5$ and $v \geq 3$ has more than $g(v-2)$ peripheral edges. However, the proof of this statement is too involved for this paper.

For practical situations the methods of Erdös and Sachs will provide ways to construct graphs which are close to optimal for most values of p.

5. Definitions and notation conventions

G	The graph (without loops).
N	The number of vertices in a finite graph.
A	The number of edges in a finite graph.
v	The degree of a regular graph. A graph is *regular* when every vertex is incident with the same number of edges. The number of edges incident with a vertex is the *degree*.
p	The probability of the removal of an edge.
G_r	A *reduced graph* of G is a subgraph which is obtained by the removal of an edge collection. An *(optimal edge vertex) assignment* is a one to one function on a subcollection of the edges into the collection of vertices of a reduced graph with maximal range. An *excluded vertex* is a vertex of the graph which is not in the range of the considered assignment.
N_r	The number of excluded vertices of the graph.
$\mathcal{E}$	The *expectation* of a stochastic function.
$F_G(p)$	The expected fraction of the vertices in the graph which is not contained in the assignment. It will be called the *service polynomial.*
$F(p)$	The *service function* of an infinite regular tree.
q	$1-p= q$. The probability that an edge is not removed.
$F^*(q)$	$F^*(q) = F(1-q)$. ; $F_G^*(q) = F_G(1-q)$. A *peripheral edge* of a subgraph S of G is an edge in G which is not in S but which is incident with a vertex of S. A *component* of a graph is a maximal connected subgraph. A *cycle* is a finite sequence of edges in which each edge has one of its vertices in common with its predecessor and the other with its successing edge. The last edge has one of its vertices in common with the one but last edge and the other with the first; the first edge has one vertex in common with the second edge and the other with the last edge.

	A *circuit* is a minimal cycle.
	A *tree* is a connected graph without cycles.
	A *spanning tree* of a graph is a reduced graph which is a tree.
n or $n(T)$	The number of edges of a tree T.
$m(T)$	The number of peripheral edges to a tree T.
	An *n-tree* is a tree with n edges.
T	A tree in a graph.
B_n	The number of n-trees incident with a vertex in an infinite regular tree.
g	The *girth* of a graph, this is the number of edges in the smallest circuit.
z	The variable representing $p^{v-2}q$.
y	The solution of the equation $y(1-y)^{v-2} = p^{v-2}(1-p)$ with $0 \leq y \leq \frac{1}{v-1}$.

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